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On Approximate Inertial Manifolds for Stochastic Navier–Stokes Equations

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The stochastic approximate inertial manifold is constructed for 2D Navier–Stokes equations with random initial data and excited by additive white noise. We estimate the size of the corresponding attracting neighborhood for this manifold. Our consideration relies on some additional information on the nonlinear term in Navier–Stokes equations.

INTRODUCTION

It is known (see [14, 24, 30, 10, 15, 5] and the references therein) that inertial manifolds (IM) are of importance in the study of the long-time behavior of trajectories of infinite-dimensional dissipative dynamical systems described by nonlinear partial differential equations. These manifolds, when they exist, are finite-dimensional invariant Lipschitz surfaces in a phase space of the system. They contain the universal attractor and attract exponentially all the trajectories. The dynamics restricted to the IM lead to a finite-dimensional ordinary differential equation which completely describes the long-time behavior of the starting system. Recently IMs have been constructed for a set of semilinear parabolic equations subjected to additive white noise [2, 8, 9]. However, the assumptions, which guarantee the existence of IM, are rather strong. That is why the concept of approximate IM has been introduced [13]. This manifold is a finite-dimensional smooth surface in a phase space, whose small vicinity attracts all the trajectories. We note also that this concept leads to new approaches to numerical investigation of long-time behavior of solutions (see, e.g., [27, 17, 12] and the literature cited there). The latter circumstance is one of the causes of

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a wide-spread interest to the study of approximate IMs. In recent years the approximate IMs have been constructed not only for a wide class of parabolic partial differential equations (see [13, 15, 31–33, 25, 26, 17, 28, 11, 5] and the references therein) but also for some classes of second-order evolution equations arising in nonlinear oscillation theory of infinite-dimensional systems [6, 7] and for nonautonomous 2D Navier–Stokes equations [18].

In this paper we consider 2D Navier–Stokes (NS) equations under periodic boundary conditions, with random initial data and excited by additive white noise. The study of this problem is connected with the statistical approach [16] in turbulence and has a long history (see, e.g., [3, 34, 19] and the references therein).

We introduce a slight modification of Foias–Manley–Temam (FMT) approximate IM (see [13]) and prove that in the mean the trajectories of the stochastic system considered are attracted closer to this FMT approximate IM than to a flat one. We estimate the size of the corresponding attracting neighborhood for this approximate IM. Lemma 1.1 is of prime importance in the proof of these estimates. This lemma contains some additional estimates of the nonlinear term in NS equations.

The paper is organized as follows. In Section 1 we introduce some notations, recall certain known facts, and prove the main Lemma 1.1. In Section 2 we recall some results on random processes and the stochastic NS equations. In Section 3 we establish some additional properties of random trajectories of the considered system. The main result of this section is Theorem 3.1 on the estimate of small eddies. Section 4 is devoted to the construction of approximate IM.

1. Notations and Preliminaries

We consider the 2D Navier–Stokes equations for a viscous incompressible fluid filling a region $T^2 = (0, L_1) \times (0, L_2)$ under periodic boundary conditions,

$$\begin{gathered} \partial_t u - \nu \Delta u + (u, \nabla)u + \nabla p = F(x,t), \qquad x \in T^2, t > 0, \\ \nabla u = 0, \qquad x \in T^2, \qquad u(x,0) = u_0(x), \end{gathered} \tag{1.1}$$

where the velocity vector $u(x, t) = (u_1(x, t); u_2(x, t))$ and the pressure $p(x, t)$ are periodic functions of period L_i in the direction x_i, $i = 1, 2$. These quantities are the unknowns. In (1.1) $\nu > 0$ is the kinematic viscosity and $F(x, t)$ represents the external body forces.

It is well known (see, e.g., [22, 29]) that this system can be written as a nonlinear evolution equation in a certain Hilbert space. Let us introduce the necessary definitions. We denote by $\mathcal{V}$ the space of trigonometric polynomials $v(x)$ of period L_i in the direction x_i, $i = 1, 2$, with values in $\mathbf{R}^2$ such that div $v = 0$ and $\int_{T^2} v(x)\, dx = 0$. We also set

$$H = \text{closure of } \mathcal{V} \text{ in } L^2(T^2) \times L^2(T^2),$$
$$V = \text{closure of } \mathcal{V} \text{ in } H^1(T^2) \times H^1(T^2),$$

where $H^s(T^2)$ is the Sobolev space of order s. For $u \in H$ and $v \in V$ we denote by

$$|u| = \left\{\int_{T^2} |u(x)|^2\, dx\right\}^{1/2} \qquad \text{and} \qquad \|v\| = \left\{\int_{T^2} |\nabla v(x)|^2\, dx\right\}^{1/2}$$

the norms in H and V, respectively. The corresponding inner product will be denoted by $(\cdot, \cdot)$ and $((\cdot, \cdot))$. Let Π be the orthogonal projection of $L^2(T^2) \times L^2(T^2)$ onto H. We define the Stokes operator $A = -\nu \Pi \Delta u = -\nu \Delta u$ and the bilinear operator $B(u, v) = \Pi(u, \nabla)v$ for all u and v lying in $D(A) = V \cap \{H^2(T^2) \times H^2(T^2)\}$. We recall (see, e.g., [21, 22, 29]) that the operator A is a positive self-adjoint operator with a discrete spectrum; i.e., there exists a complete orthonormal set $\{e_j\}$ of eigenfunctions of A such that $Ae_j = \lambda_j e_j$ and $0 < \lambda_1 \le \lambda_2 \le \ldots$. This property of the operator A implies that the following estimates hold (see, e.g., [14, 30, 5]),

$$|A^\beta e^{-tA} Q_N|_{L(H,H)} \le \left(\frac{\beta}{t} + \lambda_{N+1}\right)^\beta \exp(-t\lambda_{N+1}), \qquad \beta > 0, \tag{1.2}$$

and

$$\int_s^t |A^\beta e^{-(t-\tau)A} Q_N|_{L(H,H)}\, d\tau \le C\lambda_{N+1}^{-1+\beta}, \qquad 0 \le \beta < 1, \tag{1.3}$$

where $Q_N = I - P_N$ and P_N is the orthogonal projector in H onto the space spanned by the first N eigenfunctions of Stokes operator A. Here $|\cdot|_{L(H,H)}$ is the operator norm. We also note that $D(A^{1/2}) = V$ and $\|v\|^2 = \nu^{-1} |A^{1/2}v|^2$.

As for the bilinear operator $B(u, v)$, it is a continuous mapping from $D(A) \times D(A)$ into H and from $V \times V$ into V', where V' is the adjoint to V relative to H (see, e.g., [22, 29]). For other properties of $B(u, v)$ the reader is referred to [21, 22, 29], for example. We suppose also that $B(u) = B(u, u)$.

Now we can rewrite the system (1.1) as follows

$$\partial_t u + Au + B(u) = \Pi F(t), \qquad u_{|t=0} = u_0. \tag{1.4}$$

For results concerning the existence, uniqueness, and regularity of solutions to the deterministic NS equations (1.4) we refer the reader to [21, 22, 29] and the references therein.

The following lemma is of prime importance in our subsequent considerations.

LEMMA 1.1. *Let $0 \le \beta \le 1/2$. Then*

$$|A^\beta B(u, v)| \le C\,|A^{1/2+\delta}u| \cdot |A^{1/2+\beta}v|, \tag{1.5}$$

for $u, v \in D(A)$, where δ is any positive number and the constant $C > 0$ depends on δ and β.

Proof. Since the operator Π is the projection from $H^{2s}(T^2) \times H^{2s}(T^2)$ into

$$D(A^s) = H \cap (H^{2s}(T^2) \times H^{2s}(T^2))$$

it is clear that (1.5) follows from the inequality

$$\|f\cdot g\|_{H^s(\mathbb{O})} \le C\cdot \|f\|_{H^s(\mathbb{O})}\cdot \|g\|_{H^{1+\delta}(\mathbb{O})}, \qquad 0 \le s \le 1,\, \delta > 0, \tag{1.6}$$

where $\mathbb{O} = T^2$ and f and h lie in $\mathcal{V}$. Using the procedure of continuation (see, e.g., [23]) one can easily show that (1.6) follows from the same inequalities for $\mathbb{O} = \mathbf{R}^2$ and for f and h lying in $C_0^\infty(\mathbf{R}^2)$. Let us prove (1.6) for $\mathbb{O} = \mathbf{R}^2$. For $s = 0$ and $s = 1$ this estimate follows from the continuity of the imbedding $H^{1+\delta}(\mathbf{R}^2)$ into $L^\infty(\mathbf{R}^2)$ [4]. For the case $s = 1$ we also use the imbeddings (see [4]):

$$H^1(\mathbb{O}) \subset L^p(\mathbb{O}) \quad \text{for} \quad p > 1 \quad \text{and} \quad H^{1-\sigma}(\mathbb{O}) \subset L^{2/\sigma}(\mathbb{O}) \quad \text{for} \quad 0 < \sigma \le 1.$$

It is known [4, 23] that the norm in $H^s(\mathbf{R}^2)$ for $0 < s < 1$ can be defined by the equality

$$\|g\|_s^2 = \|g\|_{L^2(\mathbf{R}^2)}^2 + \int_{\mathbf{R}^2} \frac{dy}{|y|^{2+2s}} \int_{\mathbf{R}^2} |g(x+y) - g(x)|^2\, dx.$$

Therefore the Holder inequality yields

$$\|f\cdot h\|_s \leq \|f\cdot h\|_{L^2} + \|f\|_s \cdot \|h\|_{L_\infty} + \|f\|_{L^{2/(1-s)}} \cdot \|h\|_{s,2/s}, \tag{1.7}$$

where

$$\|h\|_{s,p} = \|h\|_{L^p} + \left(\int_{\mathbf{R}^2} \frac{dy}{|y|^{2+2s}} \left(\int_{\mathbf{R}^2} |h(x+y) - h(x)|^p \, dx\right)^{2/p}\right)^{1/2}$$

is the norm in the Besov space $B^s_{p,2}(\mathbf{R}^2)$. Consequently, using the continuity of the imbedding (see [4]) of $H^\sigma(\mathbf{R}^2)$, in the Besov space $B^\alpha_{p,2}(\mathbf{R}^2)$, for $\sigma = 1 + \alpha - 2/p$, $p > 2$, we conclude that $\|h\|_{s,2/s} \leq \| h \|_{H^1(\mathbf{R}^2)}$. Thus using the continuity of the imbeddings [4] $H^{1+\delta}(\mathbf{R}^2) \subset L^\infty(\mathbf{R}^2)$ and $H^1(\mathbf{R}^2) \subset L^p(\mathbf{R}^2)$, $p \geq 1$, we derive (1.6) from (1.7).

2. On the Stochastic Navier–Stokes Equations

Now we recall some notation and results concerning random processes and stochastically excited NS equations. Let $W(t)$ be the Wiener process in H with the correlation operator K ($K = K^* > 0$, $\operatorname{tr} K < \infty$). This means [1, 20] that there exist a probability measure space $(\Omega, \mathcal{B}, \mathcal{P})$ and Gaussian process $W(t, \omega)$, $\omega \in \Omega$, $t > 0$ on it with values in H such that (i) the process $W(t)$ has independent increments $W(t + s) - W(s)$ and $W(0) = 0$; (ii) $\mathbf{E}(W(\tau_0), h_0) = 0$ and

$$\mathbf{E}(W(\tau_1), h_1)\cdot(W(\tau_2), h_2) = \min(\tau_1, \tau_2)(Kh_1, h_2),$$

where $\tau_j \geq 0$, $h_j \in H$, $j = 0, 1, 2$. Here and below $\mathbf{E}$ is the expectation on the probability space $(\Omega, \mathcal{B}, \mathcal{P})$. For simplicity we also suppose that the correlation operator possesses the properties:

$$Ke_j = k_j e_j,\ k_j \geq 0 \qquad \text{and} \qquad \operatorname{tr} KA = \sum_{j=1}^{\infty} k_j \lambda_j < \infty. \tag{2.1}$$

It is clear [1, 20] that under these conditions $W(t, \omega)$ lies in $C(\mathbf{R}_+; V)$ almost surely. Here and below $C(\mathcal{I}, \mathcal{H})$ is the space of continuous functions on interval $\mathcal{I}$ with values in $\mathcal{H}$. Using the notion of the operator stochastic integral (see, e.g., [1]) we can also define the Gaussian process of the form

$$\xi(t, s) = \int_s^t \exp(-(t-\tau)A)\, dW(\tau), \qquad t > s \geq 0. \tag{2.2}$$

By standard methods [1] one can verify that $\xi(t, s)$ has its values in $D(A)$ and for $0 \le \beta \le 1$ possesses the properties:

(i) $\mathbf{E}A^{\beta}\xi(t,s) = 0$ and (ii)

$$\mathbf{E} \mid QA^{\beta}\xi(t,s) \mid^2 = 2\mathrm{tr}\{KA^{2\beta-1}Q(1 - e^{-2(t-s)A})\}, \tag{2.3}$$

$$\mathbf{E} \mid QA^{\beta}\xi(t,s) \mid^{2p} \le C_p\{\mathbf{E} \mid QA^{\beta}\xi(t,s) \mid^2\}^p, \qquad p \ge 0, \tag{2.4}$$

where Q is any eigen-projector of the operator A.

The main object of our consideration is NS equations (1.4) excited by random forces of the form

$$\partial_t u + Au + B(u) = f(t) + \partial_t W(t, \omega), \tag{2.5}$$

$$u_{|t=0} = u_0(\omega), \tag{2.6}$$

where $f(t)$ is a bounded function on $\mathbf{R}_+$ with values in H and $\partial_t W$ is a white noise process, i.e., generalized derivative of the Wiener process $W(t, \omega)$ with respect to t and u_0 is a random variable with values in H independent of $W(t, \omega)$, $t > 0$.

This stochastic evolution equation was investigated in [3] (see also [34, 19] and the references therein). Here we will rely on a result on the existence of strong solutions from [19].

Let $\mathcal{U}_T$ be the space of functions $u(x, t)$ lying in $C(0, T; H)$ such that

$$|u|_{\mathcal{U}_T} \equiv \sup_{0 \le s,t \le T} \frac{|u(t) - u(s)|}{|t - s|^{\delta}} + \left(\int_0^T | Au(\tau)|^2 \, d\tau\right)^{1/2} < \infty,$$

where $0 < \delta < 1/2$, $T > 0$. We also let

$$\mathcal{U} = \mathrm{proj} \lim_{T \to \infty} \mathcal{U}_T,$$

i.e., $\mathcal{U}$ is the set of vector-functions $v(t)$ on $\mathbf{R}_+$ such that $v(t)$ belongs to $\mathcal{U}_T$ for every $T > 0$. The topology in $\mathcal{U}$ is defined by collection of the seminorms $|\cdot|_{\mathcal{U}_T}$, $T > 0$.

Now we recall the result from [19] on existence of solutions of the stochastic initial problem (2.5), (2.6).

THEOREM 1.1. *Let $W(t, \omega)$ be the Wiener process in H on the probability measure space $(\Omega, \mathcal{B}, \mathcal{P})$ such that* (2.1) *holds and let $f(t)$ be a bounded continuous function from $\mathbf{R}_+$ into H. We also suppose that $u_0(\omega)$ is a* *random variable with values in V independent of $W(t, \omega)$, $t > 0$, and such that*

$$\mathbf{E}\exp(\alpha \|u_0\|^2) < \infty \tag{2.7}$$

for some $\alpha > 0$. Here and below $\mathbf{E}$ is the expectation on probability measure space $(\Omega, \mathcal{B}, \mathcal{P})$. Then there exists a unique (up to equivalence) nonanticipating random process $u(t, \omega)$, $t > 0$, which possesses the properties

(i) *$u(\cdot, \omega)$ belongs to $\mathcal{U}$ for almost every $\omega \in \Omega$;*

(ii) *the equality* (2.5) *is almost surely satisfied in the sense of generalized functions and $u(0, \omega) = u_0$ for almost every $\omega \in \Omega$;*

(iii) *there exists $\alpha_1 \in (0, \alpha)$ such that for all $t > 0$ we have*

$$\mathbf{E}\left\{\exp(\alpha_1 \cdot \|u(t)\|^2) + \int_0^t |Au(\tau)|^2 \exp(\alpha_1 \cdot \|u(\tau)\|^2)\, d\tau\right\} < \infty.$$

We note that this theorem has been proved in [19] for a time independent force f. However, the proof remains true for nonautonomous $f(t)$. One can also prove the existence and uniqueness of the solution $u(t, \omega)$ under weaker conditions on u_0 than (2.7). Here we keep assumption (2.7) for the sake of simplicity.

We also note that slightly modifying the consideration from [19] we can prove the following dissipativity properties of the solution $u(t, \omega)$:

$$\mathbf{E}\,\|u(t)\|^j \le D_1 \cdot \mathbf{E}\,\|u(0)\|^j \cdot e^{-\gamma t} + D_2, \tag{2.8}$$

and

$$\int_t^{t+a} \mathbf{E}(|Au(\tau)|^2 \cdot \|u(\tau)\|^{j-2})\, d\tau \le D_1 \cdot \mathbf{E}\,\|u(0)\|^j \cdot e^{-\gamma t} + (1 + a)D_2 \tag{2.9}$$

for certain positive constants D_i and γ. Here $t > 0$, $j \ge 2$, and $a > 0$ is fixed. The constants D_i do not depend on a. These inequalities will be useful below.

At last we note that the solution $u(t, \omega)$ of the problem (2.5), (2.6) almost surely possesses the property

$$u(t) = e^{-(t-s)A}u(s) - \int_s^t e^{-(t-\tau)A}\,(B(u(\tau)) - f(\tau))\, d\tau + \xi(t, s), \tag{2.10}$$

where $t > s \ge 0$ and $\xi(t, s)$ is defined by (2.2). This can be proved if we consider the nonlinear term in (2.5) as a known nonanticipating random process and $u(t, \omega)$ as a solution of the corresponding nonhomogeneous linear problem.

3. Additional Properties of Random Trajectories

The following assertion strengthens the dissipativity property (2.8) of the process $u(t, \omega)$.

Lemma 3.1. *Assume the hypotheses of Theorem* 1.1 *are valid. Let* $0 < \beta < 1/2$ *and* $2 < j < 1/\beta$. *Then the inequality* $\mathbf{E}\,\|u(0)\|^{2j} \leq R$ *implies that*

$$(\mathbf{E}\,|A^{1/2+\beta}u(t)\,|^{j})^{1/j} \leq C_1(R)\cdot e^{-\gamma t} + C_2 \tag{3.1}$$

for all $t \geq t^* > 0$. *Here* $\gamma > 0$ *and the constant* C_2 *is independent of* u_0.

Proof. It is clear from (2.10), (1.2), (1.3), (2.3), (2.4) that

$$\begin{aligned}(\mathbf{E}\,|A^{1/2+\beta}u(t)|^{j})^{1/j} \leq C\cdot\{&e(t-s;\beta,\lambda_1)\cdot(\mathbf{E}\,\|u(s)\|^{j})^{1/j}\\ &+\max_{\mathbf{R}_+}|f(\tau)| + (\operatorname{tr} KA^{2\beta})^{1/2}\} + n(t,s),\end{aligned} \tag{3.2}$$

where $t > s \geq 0$ and

$$e(\tau;\beta,\lambda) = \left(\frac{\beta}{\tau}+\lambda\right)^{\beta}\exp(-\tau\lambda) \tag{3.3}$$

and

$$n(t,s)^{j} = \mathbf{E}\left(\int_s^t e(t-\tau;1/2,\lambda_1)\,|\,A^{\beta}B(u(\tau)\,|\,d\tau\right)^{j}.$$

Lemma 1.1 and the interpolation inequality

$$|A^{1/2+s}u| \leq C\,|A^{1/2}u|^{1-2s}\,|Au|^{2s}, \qquad 0 \leq s \leq 1/2,$$

give that

$$|A^{\beta}B(u)| \leq C\,\|u\|^{2-2(\delta+\beta)}\,|Au|^{2(\delta+\beta)}. \tag{3.4}$$

We choose $\delta > 0$ such that $(\delta + \beta)j = 1$. Then from the Holder inequality we have

$$n(t, s)^j \leq CI(t, s) \int_s^t \mathbf{E}(|Au(\tau)|^2 \cdot \|u(\tau)\|^{2j-2})\, d\tau,$$

where

$$I(t, s) = \left\{ \int_s^t e(t - \tau; 1/2, \lambda_1)^{j/(j-1)}\, d\tau \right\}^{j-1}.$$

Since $j/(j - 1) < 2$ this integral exists. If we now suppose $s = t - t^*$ then the estimate (3.1) follows from (3.2), (2.8), and (2.9).

Let us introduce an additional assumption on the function $f(t)$. Let

$$g(t, s) = \int_s^t e^{-(t-\tau)A} f(\tau)\, d\tau. \tag{3.5}$$

We assume that $g(t, s) \in D(A)$ for all $t > s \geq 0$ and

$$L(f) \equiv \sup_{t > s \geq 0} |Ag(t, s)| < \infty. \tag{3.6}$$

We note that (3.6) is valid if for some $\theta > 0$ there exists $L > 0$ such that

$$|f(\tau) - f(t)| \leq L\, |\tau - t|^\theta. \tag{3.7}$$

In order to prove this one can use the formula

$$g(t, s) = \int_s^t e^{-(t-\tau)A} (f(\tau) - f(t))\, d\tau + A^{-1}(1 - e^{-(t-s)A}) f(t).$$

As above let $P = P_N$ be the orthogonal projection in H onto the space spanned by the first N eigenfunctions of the Stokes operator and $Q = 1 - P$. For $N = 0$ we suppose $Q = 1$.

THEOREM 3.1. *Assume the hypotheses of Theorem* 1.1 *are valid and* (3.6) *holds. Then for every $j > 1$ the inequality* $\mathbf{E}\, \|u_0\|^{4j} \leq R$ *implies the estimate*

$$(\mathbf{E}\, |Aq(t)|^j)^{1/j} \leq C_0\{L(Qf) + (\operatorname{tr} QKA)^{1/2}\} + \lambda_{N+1}^{-\sigma}(C_1(R) \cdot e^{-\gamma t} + C_2) \tag{3.8}$$

for $t > 1$. Here $q(t) = Qu(t)$, where $u(t)$ is the solution of the problem (2.5), (2.6); *σ is any positive number less than $1/2j$; $L(Qf)$ is defined by* (3.6); *the constants C_0 and C_2 do not depend on N and u_0; $\gamma > 0$.*

Proof. As in Lemma 3.1 from (2.10) and (2.3), (2.4) we have

$$(\mathbf{E}\,|Aq(t)|^j)^{1/j} \le C\{e_{1/2,N}(t-s)\,(\mathbf{E}\,\|u(s)\|^j)^{1/j} + (\operatorname{tr} QKA)^{1/2}\} + L(Qf) + n_1(t,s), \tag{3.9}$$

where $e_{\xi,N}(\tau) = e(\tau, 1-\xi, \lambda_{N+1})$ (see (3.3)) and

$$n_1(t,s)^j = \mathbf{E}\left\{\int_s^t |A^{1-\beta}e^{-(t-\tau)A}Q_N|_{L(H,H)}\,|A^\beta B(u(\tau))|\,d\tau\right\}^j.$$

Here we choose β such that $0 < \beta < 1/2$. Using (1.2) and (1.5) with $\delta = \beta$ and Holder inequality we have

$$n_1(t,s)^j \le C\left\{\int_s^t e_{\beta,N}(t-\tau)\,d\tau\right\}^{j-1} \cdot \int_s^t e_{\beta,N}(t-\tau)\mathbf{E}\,|A^{1/2+\beta}u(\tau)|^{2j}\,d\tau.$$

Let $\beta = \delta/j$, where $0 < \delta < 1/2$. Then from (1.3) and Lemma 3.1 we obtain

$$n_1(t,s) \le \lambda_{N+1}^{-\delta/j}\,(C_1(R)\cdot e^{-\gamma t} + C_2).$$

Here we suppose that $\lambda_{N+1} - 2j\gamma > (1/2)\cdot\lambda_{N+1}$. Therefore from this and (3.9) with $s = t-1$ we obtain (3.8).

From Theorem 3.1 we get immediately that $\mathbf{E}\,\|u_0\|^{4j} \le R$ implies

$$(\mathbf{E}\,|Au(t)|^j)^{1/j} \le C_1(R)\cdot e^{-\gamma t} + C_2, \qquad t \ge 1, j > 1. \tag{3.10}$$

4. Approximate Inertial Manifolds

Let

$$\psi(t) = g(t,0) + \xi(t,0), \tag{4.1}$$

where $\xi(t,s)$ and $g(t,s)$ are defined according to (2.2) and (3.5), respectively. We assume that (3.6) holds and we define time-dependent random approximate IMs as follows,

$$\mathcal{M}_t^{(i)} = \{p + \Phi_t^{(i)}(p) : p \in PH\}, \qquad i = 1,2,$$

where $\Phi_t^{(1)}(p) \equiv \Phi_t^{(1)} = Q\psi(t)$ and

$$\Phi_t^{(2)}(p) = -\int_0^t e^{-(t-\tau)A} QB(p + \psi(\tau))\, d\tau + Q\psi(t). \tag{4.2}$$

It is clear that $\Phi_t^{(1)}(p)$ and $\Phi_t^{(2)}(p)$ are time-dependent random functions from PH into $QD(A)$. We note that $\mathcal{M}_t^{(1)}$ is a time-dependent random shift of the flat manifold $\mathcal{M}^{(0)} = \{p : q = 0\}$ and the manifold $\mathcal{M}_t^{(2)}(p)$ is a slight random modification of FMT approximate IM (see [13]). We note also that

$$(\mathbf{E}\,|A\Phi_t^{(i)}(p)|^j)^{1/j} \le L(Qf) + C(R, K, f), \qquad i = 1, 2,$$

for any random variable $p \in PH$ such that $\mathbf{E}\,|Ap|^{2j} \le R$. And if we assume that $f(t) \equiv f^* \in H$ and $K \equiv 0$, i.e., white noise in (2.5) is absent, then it is easy to show that

$$(\mathbf{E}\,|A\Phi_t^{(i)}(p)|^j)^{1/j} \ge (1/2) \cdot |Qf|, \qquad i = 1, 2,$$

for $t \ge 1$ and N sufficiently large.

The following theorem gives the estimates of the mean distances between trajectories of the system (2.5), (2.6) and the manifolds $\mathcal{M}_t^{(1)}(p)$ and $\mathcal{M}_t^{(2)}(p)$ (the corresponding estimate for the flat IM $\mathcal{M}^{(0)}$ is contained in Theorem 3.1).

THEOREM 4.1. *Assume that the conditions of the Theorem* 1.1 *and* (3.6) *hold. Let* $\mathbf{E}\,\|u_0\|^{16j} \le R$ *for some* $j \ge 1$. *Then there exist* $t^* = t^*(R)$ *such that*

$$(\mathbf{E}\,|A[q(t) - Q\psi(t)]|^j)^{1/j} \le C_1 \lambda_{N+1}^{-1/2} \tag{4.3}$$

and

$$(\mathbf{E}\,|A[q(t) - \Phi_t^{(2)}(p(t) - P\psi(t))]|^j)^{1/j} \le C_2 \lambda_{N+1}^{-1} \left(1 + \log \frac{\lambda_N}{\lambda_1}\right) \tag{4.4}$$

for $t \ge t^*$. *Here* $q(t) = Qu(t)$ *and* $p(t) = Pu(t)$, *where* $u(t)$ *is the solution of the problem* (2.5), (2.6); *the quantities* $\psi(t)$ *and* $\Phi_t^{(2)}(p)$ *are defined according to* (4.1) *and* (4.2); *the constants* C_1 *and* C_2 *do not depend on* N *and* R.

Proof. We first prove (4.3). Using (2.10), (2.2), and (3.5) one can easily verify that the quantity

$$\chi(t) = u(t) - g(t, 0) - \xi(t, 0) \equiv u(t) - \psi(t) \tag{4.5}$$

satisfies equality

$$\chi(t) = e^{-(t-s)A}\chi(s) - \int_s^t e^{-(t-\tau)A} B(u(\tau))\, d\tau \tag{4.6}$$

for $t > s \geq 0$. Hence from (1.2) and (1.5) with $\beta = \delta = 1/2$ we have

$$\begin{aligned} |A\chi_1(t)| &\leq \exp(-(t-s)\lambda_{N+1})\, |A\chi_1(s)| \\ &\quad + c\int_s^t e(t-\tau; 1/2, \lambda_{N+1})\, |Au(\tau)|^2\, d\tau, \end{aligned}$$

where $\chi_1(t) = Q\chi(t)$ and $e(\tau; \beta, \lambda)$ is defined by (3.3). As above from this and (1.3), (3.10) we obtain

$$(\mathbf{E}\, |A\chi_1(t)|^j)^{1/j} = \exp(-(t-s)\lambda_{N+1})\, (\mathbf{E}\, |A\chi_1(s)|^j)^{1/j} + c\lambda_{N+1}^{-1/2}$$

for $t > s$, where s large enough. This inequality implies (4.3).

The proof of (4.4) relies on the following lemmas.

LEMMA 4.1. *Let p_1 and p_2 be random variables with values in PH such that $\mathbf{E}\, |Ap_i|^{2j} \leq R$. Then*

$$\begin{aligned} &(\mathbf{E}\, |A(\Phi_t^{(2)}(p_1) - \Phi_t^{(2)}(p_2))|^j)^{1/j} \\ &\quad \leq C_R \lambda_{N+1}^{-1/2} (\mathbf{E}\, |A(p_1 - p_2)|^{2j})^{1/2j}. \end{aligned} \tag{4.7}$$

Proof. It is clear from (1.5) with $\beta = \delta = 1/2$ and from (2.3), (2.4), (3.6) that

$$\mathbf{E}\, |A^{1/2}(B(r_1(\tau)) - B(r_2(\tau)))|^j \leq C_R\, (\mathbf{E}\, |A(p_1 - p_2)|^{2j})^{1/2},$$

where $r_k(t) = p_k + \psi(t)$, $k = 1, 2$. Consequently using the same argument as in Section 3 from (4.2) we obtain (4.7).

The following assertion contains certain continuity properties of the stochastic trajectory $u(t)$.

LEMMA 4.2. *Let $\chi(t)$ be defined by (4.5) and let $\mathbf{E}\,|u_0|^{8m} \le R$ for some $m > 1$. Then*

$$(\mathbf{E}\,|A^{1/2}P(\chi(t) - \chi(s))|^m)^{1/m} \le C\left(1 + \log\frac{\lambda_N}{\lambda_1}\right)|t - s|^{1/2} \tag{4.8}$$

for $t > s > 0$, where s is sufficiently large and C do not depend on R.

Proof. From (4.6) and (1.2) we have

$$\begin{aligned}|A^{3/2}P\chi(t)| &\le e(t - s; 1/2, \lambda_1)\,|A\chi(s)| \\ &\quad + \int_s^t |PAe^{-(t-\tau)A}|_{L(H,H)}\,|A^{1/2}B(u(\tau))|\,d\tau,\end{aligned}$$

where $e(\tau; \sigma, \lambda)$ is determined by (3.3). Let $s = t - 1$. As above, from this, (1.5) with $\delta = \beta = 1/2$, and (3.10) we obtain

$$(\mathbf{E}\,|A^{3/2}P\chi(t)|^m)^{1/m} \le C\left(1 + \int_s^t |PAe^{-(t-\tau)A}|_{L(H,H)}\,d\tau\right)$$

for t sufficiently large. By simple computation we get

$$(\mathbf{E}\,|A^{3/2}P\chi(t)|^m)^{1/m} \le C\left(1 + \log\frac{\lambda_N}{\lambda_1}\right) \tag{4.9}$$

for t sufficiently large. In the same way using (4.6) and the inequality

$$|A^{-1/2}(1 - e^{-(t-\tau)A})|_{L(H,H)} \le C\,|t - s|^{1/2},$$

we have

$$\begin{aligned}|AP(\chi(t) - \chi(s))| &\le C\,|t - s|^{1/2}\,|A^{3/2}P\chi(s)| \\ &\quad + C\int_s^t e(t - \tau; 1/2, \lambda_1)\,|Au(\tau)|^2\,d\tau.\end{aligned}$$

Using (4.9), (3.10), and the Holder inequality we derive (4.8).

Let $\chi_2(t) = q(t) - \Phi_t^2(P\chi(t))$. Then using (2.10) and the identity

$$\begin{aligned}\Phi_t^{(2)}(p) = \exp(-(t-s)A)\Phi_s^{(2)}(p) + Q(\xi(t,s) + g(t,s)) \\ - \int_s^t e^{-(t-\tau)A} QB(p + \psi(\tau))\, d\tau,\end{aligned}$$

for $p = P\chi(s)$ we have that

$$\begin{aligned}\chi_2(t) = \exp(-(t-s)A)\chi_2(s) + \Phi_t^{(2)}(P\chi(t)) - \Phi_t^{(2)}(P\chi(s)) \\ - \int_s^t e^{-(t-\tau)A} Q(B(u(\tau)) - B(P\chi(s) + \psi(\tau)))\, d\tau.\end{aligned} \tag{4.10}$$

Lemmas 4.1 and 4.2 imply that

$$\begin{aligned}(\mathbf{E}\,|A[\Phi_t^{(2)}(P\chi(t)) - \Phi_t^{(2)}(P\chi(s))]|^j)^{1/j} \\ \leq C\lambda_{N+1}^{-1/2}\left(1 + \log\frac{\lambda_N}{\lambda_1}\right)|t-s|^{1/2},\end{aligned} \tag{4.11}$$

if $\mathbf{E}\,\|u_0\|^{16j} \leq R$ and $t > s > 0$, where s is sufficiently large. Since $u(\tau) = \chi(\tau) + \psi(\tau)$, Lemma 1.1 implies that

$$\begin{aligned}(\mathbf{E}\,|A^{1/2}\{B(u(\tau)) - B(P\chi(s) + \psi(\tau))\}|^j)^{1/j} \\ \leq C\cdot\{(\mathbf{E}\,|AP(\chi(\tau) - \chi(s))|^{2j})^{1/2j} + (\mathbf{E}\,|AQ\chi(\tau)|^{2j})^{1/2j}\}\end{aligned} \tag{4.12}$$

for $\tau > s$ and s sufficiently large. Consequently using (4.10)–(4.12) and (4.3) we can obtain

$$\begin{aligned}d(t) \leq \exp(-(t-s)\lambda_{N+1}\, d(s) \\ + C_1\lambda_{N+1}^{-1/2}\left(1 + \log\frac{\lambda_N}{\lambda_1}\right)|t-s|^{1/2} + C_2\lambda_{N+1}^{-1}\end{aligned} \tag{4.13}$$

for $t > s \geq t^*$, where

$$d(t) = (\mathbf{E}\,|A\chi_2(t)|^j)^{1/j}$$

and t^* is large enough. Let $t = t^* + nh$ and $s = t^* + (n - 1)h$, where $h = 1/\lambda_{N+1}$ and $n = 1, 2, \ldots$. Then inserting these t and s into (4.13) after iterating we obtain

$$d(t_n) \leq d(t^*)e^{-n} + C\lambda_{N+1}^{-1}\left(1 + \log\frac{\lambda_N}{\lambda_1}\right). \tag{4.14}$$

It can be easily seen that (4.4) follows from (4.13) and (4.14).

In conclusion we note that in the deterministic case ($K \equiv 0$) the approximate IMs $\mathcal{M}_t^{(1)}(p)$ and $\mathcal{M}_t^{(2)}(p)$ with N sufficiently large describe long time behavior better than the manifolds constructed in [18] for nonautonomous 2D Navier–Stokes equations. However, the corresponding estimates in [18] involve the norm in the space V only.

We note also that our results remain true for other boundary conditions and for some other classes of parabolic equations.

References

1. Ja. I. Belopolskaya and Yu. L. Dalecky, "Stochastic Equations and Differential Geometry," Kluwer Academic, Dordrecht, 1990.
2. A. Bensoussan and F. Flandoli, Stochastic inertial manifold, Scuola Normale Superiore, Pisa, preprint No. 07, 1994.
3. A. Bensoussan and R. Temam, Equations stochastiques de type Navier–Stokes, *J. Funct. Anal.* **13** (1973), 195–222.
4. J. Bergh and J. Lofstrom, "Interpolation Spaces: An Introduction," Springer-Verlag, New York, 1976.
5. I. D. Chueshov, "Introduction in Theory of Inertial Manifolds, Lecture Notes," Kharkov Univ. Press, Kharkov, 1992. [Russian]
6. I. D. Chueshov, Approximate inertial manifolds for second order evolution equation, *Dokl. Acad. Sci. Ukraine* **3** (1993), 42–45.
7. I. D. Chueshov, Regularity of solutions and approximate inertial manifolds for von Karman evolution equations, *Math. Methods Appl. Sci.* **17** (1994), 667–680.
8. I. D. Chueshov and T. V. Girya, Inertial manifolds for stochastic dissipative dynamical systems, *Dokl. Acad. Sci. Ukraine* **7** (1994), 42–45.
9. I. D. Chueshov and T. V. Girya, Inertial manifolds and invariant measures for semilinear parabolic equations subjected to additive white noise, Forshungszentrum BiBoS, Universität Bielefeld, preprint No. 657/7/94, 1994.
10. P. Constantin, C. Foias, B. Nicolaenko, and R. Temam, "Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations," Springer-Verlag, New York, 1989.
11. A. Debussche and M. Marion, On the construction of families of approximate inertial manifolds, *J. Differential Equations* **100** (1992), 173–201.
12. C. Devulder, M. Marion, and E. Titi, On the rate of convergence of the nonlinear Galerkin methods, *Math. Comp.* **60** (1993), 495–514.

13. C. Foias, O. Manley, and R. Temam, Modeling of the interaction of small and large eddies in two dimensional turbulent flows, *Math. Mod. Numer. Anal.* **22** (1988), 93–114.
14. C. Foias, G. R. Sell, and R. Temam, Inertial manifolds for nonlinear evolutionary equations, *J. Differential Equations* **73** (1988), 309–353.
15. C. Foias, G. R. Sell, and E. Titi, Exponential tracking and approximation of inertial manifolds for dissipative equations, *J. Dynamics Differential Equations* **1** (1989), 199–224.
16. E. Hopf, Statistical hydrodynamics and functional calculus, *J. Rational Mech. Anal.* **16** (1948), 87–123.
17. M. S. Jolly, I. G. Kevrekidis, and E. S. Titi, Approximate inertial manifolds for the Kuramoto–Sivashinsky equation: Analysis and computations, *Phys. D* **44** (1990), 38–60.
18. D. A. Jones and E. S. Titi, A remark on quasi-stationary approximate inertial manifolds for the Navier–Stokes equations, *SIAM J. Math. Anal.* **25** (1994), 894–914.
19. A. I. Komech and M. I. Vishik, Strong solutions of 2D Navier–Stokes system and the corresponding Kolmogorov equations, *Z. Anal. Anwendungen* **1,** 3 (1982), 23–52. [Russian]
20. H. H. Kuo, "Gaussian Measures in Banach Spaces," Springer-Verlag, New York, 1972.
21. O. A. Ladyzhenskaya, "The Mathematical Theory of Viscous Incompressible Flow," 2nd ed., Gordon & Breach, New York, 1969.
22. J.-L. Lions, "Quelques Methodes de Resolution de Problemes aus Limites non Lineaires," Dunod, Paris, 1969.
23. J.-L. Lions and E. Magenes, "Non-Homogeneous Boundary Value Problems and Applications," Vol. 1, Springer-Verlag, New York, 1972.
24. J. Mallet-Paret and G. R. Sell, Inertial manifolds for reaction diffusion equations in higher space dimensions, *J. Amer. Math. Soc.* **1** (1988), 805–866.
25. M. Marion, Approximate inertial manifolds for the pattern formation Cahn–Hillard equation, *Math. Mod. Numer. Anal.* **23** (1989), 463–488.
26. M. Marion, Approximate inertial manifolds for reaction diffusion equations in high dimension, *J. Dynamics Differential Equations* **1** (1989), 245–267.
27. M. Marion and R. Temam, Nonlinear Galerkin methods, *SIAM J. Numer. Anal.* **26** (1989), 1139–1157.
28. K. Promislow, Induced trajectories and approximate inertial manifolds for the Ginzburg–Landau partial differential equation, *Phys. D* **41** (1990), 232–259.
29. R. Temam, "Navier–Stokes Equations, Theory and Numerical Analysis," 3rd ed., North-Holland, Amsterdam, 1984.
30. R. Temam, "Infinite Dimensional Dynamical Systems in Mechanics and Physics," Springer-Verlag, New York, 1988.
31. R. Temam, Induced trajectories and approximate inertial manifolds, *Math. Mod. Numer. Anal.* **23** (1989), 541–561.
32. R. Temam, Attractors for the Navier–Stokes equations: Localization and approximation, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **36** (1989), 629–647.
33. E. S. Titi, On approximate inertial manifolds to the Navier–Stokes equations, *J. Math. Anal. Appl.* **149** (1990), 540–557.
34. M. I. Vishik and A. V. Fursikov, "Mathematical Problems of Statistical Hydromechanics," Kluwer Academic, Dordrecht, 1988.